Viscosity Prescriptions in Accretion Disks with Shock Waves

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Abstract

We study the evolution of viscous, isothermal, rotating, thin, axisymmetric accretion disks around a compact object using Smoothed Particle Hydrodynamics. We emphasize the effects of different choices of viscosity prescriptions on the evolution of angular momentum as well as other physical quantities of the disk. We show that a flow with the Shakura-Sunyaev viscosity prescription may produce only *shear* shock, where the angular momentum changes significantly across the shock. We present and study the effects of other prescriptions of viscosity which render the angular momentum continuous across the shock waves. In general, it is observed that for flows with a small viscosity, the shocks are weaker, form farther away and are wider as compared to the shocks in inviscid flows. If the viscosity is high, shocks do not form at all. The flow remains subsonic and Keplerian throughout the disk and becomes supersonic only very close to the horizon.

1. Introduction

A large number of numerical simulation results present in the literature suggest that shocks may form in adiabatic accretion disks (Hawley, Smarr & Wilson 1984; Chakrabarti & Molteni 1993; Molteni, Lanzafame & Chakrabarti, 1994) and in inviscid disks with radiative transfer (Molteni, 1994). These shock solutions were obtained for inviscid flows and agree with the theoretical predictions (Chakrabarti, 1989, 1990a) reasonably well. Chakrabarti (1990b, hereafter SKC90b) discussed effects of viscosity on the shock formation in isothermal accretion disks around black holes. The general conclusions were that the stable shock (X_{s3} in the notation of SKC90b) is weaker and forms farther away as the viscosity parameter is increased. When the viscosity is very high, shocks do not form at all. Instead, the flow remains subsonic and Keplerian throughout the disk and becomes supersonic close to the black hole after passing through the inner sonic point. In the present paper, we shall study the numerical evolution of the viscous isothermal disks with a particular emphasis on the nature of shocks in flows close to a black hole. We show that the solution strongly depends upon the viscosity prescription. In particular, we show that if the transport of angular momentum follows the Shakura-Sunyaev (1973) α viscosity (hereafter SSα) prescription, a shear shock must form, where a jump of angular momentum take place. We present other prescriptions where angular momentum is continuous across the shock waves and for high viscosity yields shock-free Keplerian disks exactly as the Shakura-Sunyaev prescription. These prescriptions, therefore, could possibly be more suitable for disks which include shock waves as well as for shock free Keplerian disks.

The plan of our paper is the following: In the next Section, we show how different viscosity prescriptions might affect the nature of shock transitions. For simplicity of arguments we assume only the *isothermal* flows. In Section 3, we present a test of our code to show that with a significant viscosity, the disk does become Keplerian. Subsequently, we present a few numerical solutions of the isothermal flows with various viscosity prescriptions and compare these results. Finally, in Section 4, we summarize

our results and draw conclusions.

2. Model Equations with Different Viscosity Prescriptions

Consider a thin, isothermal, axisymmetric, accretion flow onto a compact object. The radial momentum equation is:

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{1}{\Sigma}\frac{\partial W}{\partial r} - \frac{\lambda^2}{r^3} + \frac{\partial \Phi}{\partial r} = 0. \tag{1a}$$

The continuity equation is given by,

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\Sigma r v_r) = 0 \tag{1b}$$

The azimuthal momentum equation is given by,

$$\frac{\partial \lambda}{\partial t} + v_r \frac{\partial \lambda}{\partial r} = \frac{1}{r\Sigma}\frac{\partial}{\partial r}(r^2 W_{r\phi}) \tag{1c}$$

Here, v_r and λ are the radial velocity and the azimuthal angular momentum respectively, and W, Σ and $W_{r\phi}$ are the pressure, density and the $r\phi$ component of the viscous stress tensor respectively. $\Phi(r,\theta)$ is the gravitational potential of the central object. This may be any of the Newtonian or pseudo-Newtonian potentials. For example, $\Phi = -1/2(r-1)$ (Paczyński & Wiita, 1980) could be used for flows around a Schwarzschild black hole or a more complex form (Chakrabarti & Khanna 1992) for flows around a Kerr black hole. All distances, velocities and time scales are measured in units of $2GM/c^2$, c and $2GM/c^3$ respectively.

If we consider the solution to be steady, the time derivatives of the above equations vanish and we get the following conservation equations,

(a) Conservation of energy:

$$\mathcal{E} = \frac{1}{2}v_r^2 + K^2 log(\Sigma) + \frac{1}{2}\frac{\lambda^2}{r^2} + \Phi \tag{2a}$$

(b) Conservation of baryons,

$$\dot{M} = \Sigma v_r r \tag{2b}$$

and,

(c) Conservation of the angular momentum

$$\dot{M}(\lambda - \lambda_e) = -r^2 W_{r\phi} \tag{2c}$$

Here, we have used the polytropic equation of state $W = const. \, \Sigma$ appropriate for isothermal flows and $K = (W/\Sigma)^{1/2}$ is the isothermal sound speed. Here, λ_e is the angular momentum at the inner edge of the disk.

A black hole accretion is necessarily transonic (Chakrabarti 1989, 1990ab). An inviscid flow may be Bondi-like, i.e., simply passes through a sonic horizon and falls onto a black hole supersonically. However, as Liang and Thomson (1980) pointed out, the flow may have more than one saddle type sonic points. Subsequently, is it shown that standing shocks may be developed in rotating winds and accretion (Habbal & Tsinganos 1983, Ferrari et al. 1985, Fukue 1987, Chakrabarti 1989, 1990ab). In particular, Chakrabarti (1990b) showed that the topological properties of the phase space behaviour of the flow depend strongly upon the angular momentum distribution and therefore on the nature of viscosity.

In an inviscid, axisymmetric flow ($W_{r\phi} = 0$), the characteristics (such as the location, strength, etc.) of the shock is determined by the shock conditions (Chakrabarti, 1990ab),

(a) Temperature of the flow is constant across the shock:

$$K_- = K_+ \tag{3a}$$

(b) Baryon flux is conserved,

$$\dot{M}_- = \dot{M}_+ \tag{3b}$$

(c) The total pressure is balanced,

$$W_- + \Sigma_- v_{r-}^2 = W_+ + \Sigma_+ v_{r+}^2 \tag{3c}$$

and,

(d) Angular momentum flux is conserved,

$$\lambda_- = \lambda_+ \tag{3d}$$

Here, $-$ and $+$ signs represent quantities in the pre-shock and post-shock flows respectively.

In the presence of viscosity, the angular momentum is transported following Equation (2c). In the following subsections, we discuss how the transport process depends on the viscosity prescriptions.

2a. Shakura-Sunyaev Viscosity Prescription

Let us first consider the effects of SSα prescription. In this case, we have $W_{r\phi} = \alpha_s W \frac{r}{\Omega_K}\frac{d\Omega}{dr}$, and the angular momentum distribution is obtained from (SKC90b),

$$\lambda - \lambda_e = \frac{\alpha_s r^3 W}{\Omega_K \dot{M}}\frac{d\Omega}{dr} = \frac{\alpha_s r K}{M\Omega_K}\frac{d\Omega}{dr} \tag{4}$$

where, $M = v_r/K$ is ths Mach number of the flow, and λ_e is the angular momentum at the inner edge of the disk and Ω_K is the local Keplerian angular velocity. The subscript s of α is to indicate the Shakura-Sunyaev α parameter. It is evident that at the shock, since M varies from supersonic ($M_- > 1$) to subsonic ($M_+ < 1$), the rate of transport of angular momentum would be quite different on both sides of the shock. In particular, in the post shock region, the angular momentum would be 'piling' up as the preshock flow is unable to transport it away efficiently. Thus, the angular momentum must be discontinuous and λ_+ must be higher compared to λ_-. Thus the shock formed would be of 'shear-type'.

2b. Continuous Angular Momentum Prescriptions

In the case where the angular momentum is assumed to be continuous across the shock wave, which is reasonable for any axisymmetric flows away from narrow

boundaries, one cannot use SSα prescription. Continuity of λ implies continuity of the viscous stress (cf. Equation 2c) and one notes that at the shock (assumed here to be of infinitesimal width),

$$W_{r\phi-} = W_{r\phi+} \tag{5}$$

In the usual form:

$$W_{r\phi} = \nu \Sigma r \frac{\partial \Omega}{\partial r} \tag{6}$$

Here, ν is the kinematic viscosity coefficient, and $\Omega = \lambda/r^2$ is the local angular velocity. Since λ is assumed to be smooth and continuous, Ω is also a smooth and continuous function. One way to achieve the continuity of λ across the shock is to define kinematic coefficient,

$$\nu_p = \frac{\alpha_p (K^2 + v_r^2)}{\Omega_K} \tag{7a}$$

so that the viscous stress is,

$$W_{r\phi} = \frac{\alpha_p (K^2 + v_r^2) \Sigma r}{\Omega_k} \frac{\partial \Omega}{\partial r}. \tag{7b}$$

In this case, the pressure balance condition (Eqn. 3c) ensures the continuity of $W_{r\ phi}$. This prescription will be referred to as the 'pressure balanced viscosity prescription'. The subscript p distinguishes this prescription from other ones.

One could instead use the balance of the mass flux in defining the kinematic viscosity,

$$\nu_m = \frac{\alpha_m v_r}{\Omega_K} \tag{7c}$$

so that the viscous stress becomes,

$$W_{r\phi} = \alpha_m |v_r| \Sigma r \frac{\partial \Omega}{\partial r}. \tag{7d}$$

The continuity of the viscous stress follows directly from the conservation of the baryon flux (Eqn. 3b). This prescription will be referred to as 'mass flux balanced viscosity prescription'. The subscript m distinguishes this prescription from other ones.

2c. Prescription with Flux-limited Diffusion

Recently, an important issue is brought up that in the boundary layer of a slowly rotating star the SSα prescription implies that the radial flow has to be supersonic and therefore would loss the causal contact with the star surface (Narayan, 1992). In this situation, the angular momentum is transported rather rapidly and $\frac{\partial \Omega}{\partial r}$ cannot be a constant. To remedy the problem of causality, a modified form of the kinematic viscosity coefficient was prescribed:

$$\nu_d = \alpha_d K^2 (1 - v_r^2/v_t^2)^2/\Omega_K \tag{8}$$

Here, $v_t \sim \beta a$, (with β of the order unity) is the turbulent speed with which angular momentum is transported fastest. This important modification brings the disk solution in the causal contact with the star by enforcing a subsonic accretion. The subscript d distinguishes this prescription from other ones.

In the case of black hole accretion this specific problem does not arise since the inner boundary condition dictates that the flow *has to be* supersonic. Secondly, as in the case of disks in stellar systems, angular momentum transport is not always due to hydrodynamical processes. Radiative, magnetic and other means of transport processes could intervene and the prescriptions in Subsections 2a and 2b need not be modified any further. However, the causality argument seems to be very reasonable one make (though radiative viscosity can still work) although one wonders if the modification of ν_p and ν_m as given by Eqn. (7a) and Eqn. (7c) after multiplying each of them by a factor $(1 - v_r^2/v_t^2)^2$ solves the problem. This is because $W_{r\phi}$ in the preshock flow would be negligible compared to its value in the postshock flow and the shock would again become 'shear-type' and piled up angular momentum may continuously drive the shock outwards.

3. Evolution of Isothermal Viscous Accretion Disks

In Chakrabarti & Molteni (1993) and Molteni & Sponholz (1994) the basic procedures for the implementation of the Smoothed Particle Hydrodynamics in cylindrically symmetric coordinates are presented and it was shown that the simulation results agree

very well with the theoretical predictions regarding the shock parameters. In the present paper, we do not discuss this any more. The only modification over the Chakrabarti & Molteni (1993) is the addition of the angular momentum transport equation (1c) with the possibility to use various dynamical viscosity prescriptions as discussed in §2 above. As before, we use Paczyński & Wiita potential (1980) to describe the external Schwarschild geometry. We wish to note in passing that our code is tested to be free from any numerical viscosity which is shear type. This is because each of the pseudo-particles is axially symmetric and interacts as a torus.

3a. Test Results

In Chakrabarti & Molteni (1993) we already shown that a flow free from viscosity produces shock waves exactly where shock waves are predicted in disks. The code we use here has been tested for adiabatic flows in Molteni and Sponholz (1994). Here, we first show a test result to indicate that our code is good enough to produce a Keplerian isothermal disk as well. Evolution of a ring of matter into a Keplerian disk in presence of viscosity has been demonstrated long ago (e.g., Pringle, 1981). In our simulation, we inject particles at the outer edge of the disk at $r_o = 100$ with angular momentum 7.00 which is close to Keplerian value at r_o. The (constant) sound speed is chosen to be $K = 0.005$ and the injection velocity was $v_0 = 0.003$. The artificial viscosity parameters were $A = 1$ and $B = 1$ (see Chakrabarti & Molteni 1993 where these parameters are denoted as α and β). The SPH parameters were: particle size $h = 0.4$ and the particle separation $\delta_p = 0.2$ (See, e.g., Monaghan, 1992 for definitions). The viscosity slowly works on the flow and transports the angular momentum outwards, which enables matter to fall onto the central black hole. Particles reaching beyond the outer grid as well as the below $r = 1.3$ were removed. In this way particles with higher angular momentum is removed from the outer edge and the particles with lower angular momentum is swallowed by the black hole. Figs. 1(a-b) shows the angular momentum distribution achieved after flow becomes steady. Fig. 1a is the result of a simulation using SSα prescription (Eqn. 4) and Fig. 1b is the result of a simulation

using mass flux balanced viscosity prescription (Eqn. 7d). Alpha parameters chosen are $\alpha_s = \alpha_m = 0.25$. In each of these figures, there are about 500 particles. In both the cases, final distribution seems to be very close to the Keplerian distribution (dotted curves) till the marginally stable orbit at $r = 3$ after which the flow falls freely and supersonically to the black hole. The excellent agreement suggests that the code with the inclusion of viscosity is working very well.

In all the simulation results listed below, we use the following quantities: The outer edge is at $r_0 = 18$, the velocity of sound $K = 0.05$, the specific angular momentum $l = 1.89$ and the injection velocity $v_{ro} = 0.1$. Various cases are distinguished by different viscosity prescriptions.

3b. Results using Shakura-Sunyaev Prescription

In Fig. 2(a-b) we compare Mach number and angular momentum variations in viscous and inviscid flows. The viscosity parameter $\alpha_s = 0.01$ is chosen everywhere in the viscous disk simulation. Note that the shock in the viscous disk forms farther out and is weaker and wider as expected (SKC90b). Fig. 2b shows that a shear shock is formed with a jump in angular momentum at the shock ($\lambda_+ > \lambda_-$). This is because the transport rate of angular momentum is higher in the postshock flows compared to the rate in the preshock flows. The angular momentum is super-Keplerian in the postshock flow and almost constant in the preshock flow.

As the α_s parameter is increased, the flow behaviour changes dramatically. Fig. 3(a-b) shows the evolution of Mach number and angular momentum when $\alpha_s = 0.1$ is chosen. One notices that the shock rapidly propagates outwards making the entire flow outside the inner sonic point subsonic, and the angular momentum distribution resembles more and more Keplerian. Successive curves are drawn at intervals of $\Delta t = 500$. The infall time-scale is $t_i = 75$ in the same unit. What clearly happens in this case is that the piled up angular momentum in the post shock flow drives the shock

outwards continuously and no steady solution becomes possible.

The topological properties of a viscous accretion flow are discussed in detail in SKC90b which we do not repeat here. A salient point that was observed is that when viscosity is very low, a stable and an unstable shock may form (in notation of SKC90b, at X_{s3} and X_{s2} respectively) with the stable shock (X_{s3}) gradually becoming weaker as viscosity is increased. When viscosity crosses a critical value, the stable shock disappears altogether. Instead, two solutions, both coming from infinity to the horizon are seemingly allowed, none being suitable for a stable shock formation. The solution passing through the inner sonic point has a higher dissipation and entropy (Chakrabarti, 1989, 1990a) and is chosen by the realistic flow. Our results presented in Fig. 1(a-b) and Fig. 3(a-b) above seem to correspond to this branch of the solution.

3c. Results using Mass Flux Balanced Viscosity Prescription

Figure 4(a-b) shows the variation of Mach number and angular momentum profile for flows evolved with mass flux balanced viscosity prescription discussed in §2 above. There are three curves in each Figure. The solid curve is obtained with $\alpha_m = 0.01$ everywhere and the long dashed curve is obtained with $\alpha_m = 0.01$ only in the subsonic region while $\alpha_m = 10^{-6}$ in the supersonic region. This latter case was chosen to mimic the flux-limited diffusion prescription (Narayan, 1992). It is clear that the angular momentum distribution is roughly monotonic and sub-Keplerian except close to the black hole where it is super-Keplerian. The introduction of the flux-limited transport prescription changes the properties of the flow in obvious ways. The shock becomes stronger, but still located farther out. The angular momentum distribution becomes strongly non-monotonic and the shock is of 'shear-shock' type where a jump in angular momentum occurs.

3d. Results using Pressure Balanced Viscosity Prescription

Figures 5 shows the angular momentum distribution of the disk with pressure balanced viscosity prescription. Solid curve is obtained with $\alpha_p = 0.1$ everywhere in the flow and the dashed curve is obtained using $\alpha_p = 0.1$ only in the subsonic region while using $\alpha_p = 10^{-5}$ in the supersonic region. The angular momentum distribution is non-monotonic but is smooth and continuous apart from some numerical noise. The dashed curve with flux-limited transport prescription is smoother in comparison. The jump in angular momentum is smaller. Dotted curve represents the Keplerian distribution for reference purpose.

In Fig. 6, we present a comparison of the angular momentum distributions obtained using all the three prescriptions discussed above. The postshock flow shows a significant variation of the distribution, while the slopes in the supersonic preshock region remain very similar. Note that α_p is chosen to be an order of magnitude higher, since the pressure balanced prescription requires the dynamical viscosity to proportional to the square of the velocities (Eqn. 7a).

4 Concluding Remarks

In this paper, we have systematically presented numerical evolution of thin, isothermal accretion disks following various viscosity prescriptions. We discovered several significant results: we find that when viscosity is low enough, the shocks are weaker, wider and form farther out. For high viscosity, the flow does not produce a stable shock due to qualitative change in the topological property of the flow. Rather, an unstable shock travels outwards sweeping the disk and making it subsonic and Keplerian. The flow becomes supersonic only at the inner sonic point located close to the horizon. These results agree with the theoretical expectations (Chakrabarti 1990b). The axisymmetric solutions seem to be stable. In future, one needs to study if these solutions remain stable under the non-axisymmetric perturbations as well.

We make here an important observation that the Shakura-Sunyaev prescription,

regarded widely as the working description of viscosity in accretion disks, does not satisfactorily describe the shocks in the disks, since the shocks are always shear-type and the jump in the angular momentum produces a region of a negative slope (although, we not see any otherwise unpleasant behaviour in terms of stability). In search of an alternate description, it became clear that one could instead use the pressure balance condition or the mass flux conservation conditions to define viscous stress. Each of these prescriptions seems to describe the shocks better. While the pressure balanced prescription still produces a smaller jump in angular momentum at the shock, the mass flux balanced prescription gives roughly monotonic, smooth and continuous distribution of angular momentum in the disk, including regions across the shock. When the viscosity is high, this latter prescription reproduces shock-free Keplerian subsonic solution as well. Thus we believe that the mass flux balanced prescription is probably more satisfactory.

The flux-limited diffusion condition, when coupled to any of our prescriptions, produced shear shocks in the disks since the viscous stress does not remain continuous. It is possible that in reality, purely hydrodynamical processes are rare and the disk angular momentum may be transported via magnetic turbulences or radiative processes. In future, we plan to investigate these issues in more detail.

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Fig. 1(a-b): Comparison of the Keplerian angular momentum distribution (dotted) with the distribution obtained by numerical simulations (solid) using (a) Shakura-Sunyaev viscosity prescription ($\alpha_s = 0.25$) and (b) mass flux balanced viscosity prescription ($\alpha_m = 0.25$). Thick solid patches are due to denser number of particles in some regions.

Fig. 2(a-b): Comparison of (a) Mach number variation and (b) angular momentum distribution in a disk using Shakura-Sunyaev viscosity $\alpha_s = 0.01$ (solid curves) with those in an inviscid disk (dashed curves). The shock in viscous disk is wider, weaker and farther out. In (b), Keplerian angular momentum is also shown for comparison. Note the super-Keplerian region close to the hole and the angular momentum jump at the shock.

Fig. 3(a-b): Non-steady evolution of the (a) Mach number variation and (b) the angular momentum distribution in a disk with a large Shakura-Sunyaev viscosity ($\alpha_s = 0.1$). Successive curves are drawn at intervals of $\Delta t = 500$ (infall time scale $t_i = 75$). The shock propagating outwards makes the entire flow subsonic except in the vicinity of the horizon, and the distribution of angular momentum gradually becomes Keplerian. The steady inviscid disk solution is also presented for comparison.

Fig. 4(a-b): Solid curves showing (a) Mach number and (b) angular momentum distribution of a disk with mass flux balanced viscosity prescription while long dashed curves are for flows in which viscosity is suppressed artificially in the supersonic region following flux-limited diffusion prescription. $\alpha_m = 0.1$ was used in both the cases. Short dashed curves indicate solutions for inviscid flows. In (b), Keplerian distribution is provided for comparison.

Fig. 5: Angular momentum distribution of the disk with pressure balanced viscosity prescription. Solid curve is with $\alpha_p = 0.1$ everywhere in the flow and the dashed curve is with $\alpha_p = 0.1$ only in the subsonic region, but $\alpha_p = 10^{-5}$ in the supersonic region. Dotted curve represents the Keplerian distribution for comparison.

Fig. 6: Comparison of angular momentum distributions in disks with various viscosity prescriptions: with $\alpha_s = 0.01$ (long dashed curve), with $\alpha_m = 0.01$ (dotted curve) and with $\alpha_p = 0.1$ (solid, ragged). The marked variation occurs only in the postshock region. An order of magnitude higher α_p was used, since the dynamical viscosity varies as the square of the velocities. Keplerian distribution (solid, smooth) is provided for reference.